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$SO(3)$ invariants of Seifert manifolds and their algebraic integrality *

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Abstract

For Seifert manifold $M = X(p_1/q_1, p_2/q_2, \cdots, p_n/q_n)$, $\tau_r^{'}(M)$ is calculated for all r odd ≥ 3. If r is coprime to at least $n-2$ of p_k (e.g. when M is the Poincare homology sphere), it is proved that $(\sqrt{\frac{4}{r}}\sin\frac{\pi}{r})^{\nu}\tau_r^{'}(M)$ is an algebraic integer in the r-th cyclotomic field, where ν is the first Betti number of M. For the torus bundle obtained from trefoil knot with framing 0, i.e. $X_{\text{tref}}(0) = X(-2/_1, 3/_1, 6/_1)$, $\tau_r^{'}$ is obtained in a simple form if $3 \not| r$, which shows in some sense that it is impossible to generalize Ohtsuki's invariant to 3-manifolds being not rational homology spheres.

1 Introduction

Consider Seifert manifolds of the form $M = X(p_1/q_1, p_2/q_2, \cdots, p_n/q_n)$, where p_k and q_k are not zero and coprime (the case of some p_k or q_k being zero is trivial, as explained in [GF]), while $\sum_k \frac{q_k}{p_k} = 0$ (i.e. M is not a rational homology sphere) is allowed.

In the case of $\sum_k \frac{q_k}{p_k} \neq 0$, r being a prime and $r \not| p_k$, $r \not| q_k$, $k = 1, 2, \cdots, n$, Rozansky [R1] has obtained a simpler formula for $\tau_r^{'}(M)$ defined by Kirby and Melvin [KM1]. Except obtaining a general formula for all odd $r \geq 3$, we obtain simpler formula in the case of r being coprime to all p_k. And after changing Rozansky's formula, we see that it becomes our form in his case.

Now, let us introduce some notations. $P = \prod_k p_k$, $H = P\sum_k \frac{q_k}{p_k}$, $c_k = (r, p_k)$ being the commom factor, $s(q,p)$ is the Dedekind sum, and $(\frac{a}{b})$ is the Jacobi symbol for odd $b > 0$, while the ratio of c to d will be written as $\frac{c}{d}$ or $c/_d$. $p_k^*$and q_k^* are any pair satisfying

$$p_k^* p_k + q_k^* q_k = 1$$

while for $c_k = (r, p_k)$, $(p_k/c_k)'$ and $(r/c_k)'$ are any pair defined by

$$(p_k/c_k)'\frac{p_k}{c_k} + (r/c_k)'\frac{r}{c_k} = 1$$

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The functions $\varepsilon(r)$ is defined by $\varepsilon(r)=1$, if $r\equiv 1 \pmod 4$, $\varepsilon(r)=i=\sqrt{-1}$, if $r\equiv -1 \pmod 4$.

And if $c_k=1, 12s^{\surd}(q_k,p_k)\equiv(12p_k(q_k,p_k))p_k' \pmod r$. Moreover, $\chi^{k,\pm}(j)$ is defined by

$$\chi^{k,\pm}(j)=\begin{cases}\pm 1, & \text{if } j\mp q_k^*\equiv 0 \pmod{c_k}\\ 0, & \text{otherwise}\end{cases}$$

and e_a denotes $e^{\frac{2\pi i}{a}}$, $A=e_r^{\frac{1\mp r}{4}}$ for $r\equiv\pm 1 \pmod 4$. We have

Theorem 1. With the notations as above, and assume all $q_k>0, r$ odd ≥ 3, then

$$\tau_r'(X(p_1/q_1,\quad p_2/q_2,\cdots,p_n/q_n))=(\sqrt{\tfrac{1}{r}}\sin\tfrac{\pi}{r})^{1-sign|H|}(-2\varepsilon(r)\sqrt{r})^{-sign|H|}\times$$

$$(signP(-sign\frac{H}{P}+1-sign|H|))^{\frac{r+1}{2}}(A^2-A^{-2})^{-2+sign|H|}\times$$

$$A^{-3sign\frac{H}{P}+\sum_{k=1}^{n}(-12s(q_k,p_k)+\frac{q_k+q_k^*}{p_k}-(p_k/c_k)'\frac{q_k^*}{c_k}-(r/c_k)'\frac{r}{c_k}p_k^*q_k^*)}\times$$

$$\prod_{k=1}^{n}((-1)^{\frac{r-1}{2}\frac{c_k-1}{2}}\sqrt{c_k}\varepsilon(c_k)(\frac{p_k/c_k}{r/c_k})(\frac{q_k}{c_k}))\times$$

$$\sum_{j=1}^{r}(A^{2j}-A^{-2j})^{2-n}A^{-(\sum_{k=1}^{n}(p_k/c_k)'\frac{q_k}{c_k})j^2}\times$$

$$\prod_{k=1}^{n}\sum\nolimits_{\pm}\chi^{k,\pm}(j)A^{\pm 2(\frac{1}{c_k}(p_k/c_k)'+p_k^*(r/c_k)'\frac{r}{c_k})j}$$

Corollary. If r is coprime to all p_k, i.e. $c_k=1$ for $k=1,\cdots,n$, then

$$\tau_r'(X(p_1/q_1,p_2/q_2,\quad\cdots,p_n/q_n))=(\sqrt{\tfrac{1}{r}}\sin\tfrac{\pi}{r})^{1-sign|H|}(-2\varepsilon(r)\sqrt{r})^{-sign|H|}\times$$

$$(-sign\frac{H}{P}+1-sign|H|)^{\frac{r+1}{2}}(A^2-A^{-2})^{-2+sign|H|}(\frac{|P|}{r})signP\times$$

$$A^{-3sign\frac{H}{P}+P'H-12\sum_{k=1}^{n}s^{\surd}(q_k,p_k)}\times$$

$$\sum_{j=1}^{r}(A^{2j}-A^{-2j})^{2-n}A^{-P'Hj^2}\prod_{k=1}^{n}(A^{2p_k'j}-A^{-2p_k'j})$$

where $P'P\equiv 1 \pmod r$

Remark 1. In Theorem 1, the assumption of all $q_k > 0$ is necessary, otherwise the formula would be changed to a more complicated form. It can be seen from the proof. While for the corollary, we need not this assumption.

Theorem 2. If r is coprime to ar least $n-2$ of p_k, then

$$\tau_r^{'}(X(p_1/q_1, p_2/q_2, \cdots, p_n/q_n)(\sqrt{\frac{1}{r}}\sin\frac{\pi}{r})^{sign|H|-1}$$

is an algebraic interger in the r-th cyclotomic field.

Remark 2. We actually prove that $\Theta_r(M)$ and $\xi_r(M)$ defined in [BHMV] and [Li] respectively are all algebraic integers in this case.

From Theorem 2, we see that for all rational homology sphere of the form $X(2/q_1,\ p_2/q_2, p_3/q_3),\ \tau_r^{'}$ is an algebraic integer for all odd $r \geq 3$. Especially it is true for all RHS obtained by Dehn surgery on $(2,n)$ torus knot, including those considered by R. Lawsence in [La]. Interestingly the Poincare homology sphere $\sum(2,3,5)$ and all the Brieskon homology spheres of the form $\sum(2^n, p, q)$ are among them.

For the torus bundle over S^1 obtained by the monodromy matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL(2,Z)$$

which is the Seifert manifold $X(-2/_1, 3/_1, 6/_1)$ or $X_{\text{tref}}(0)$, i.e. gotten by doing surgery on left-handed trefoil knot with framing 0, we calculate $\tau_r^{'}$ further and get

$$\tau_r^{'}(X_{\text{tref}}(0)) = \begin{cases} 0, & \text{if } r \equiv 1 \pmod 3 \\ \dfrac{-\sqrt{r}}{2sin\frac{\pi}{r}}e^{-\frac{2\pi}{r}i}, & \text{if } r \equiv -1 \pmod 3 \end{cases}$$

For rational homology 3-sphere M, there is Ohtsuki's invariant

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n(M)(t-1)^n \in Q[[t-1]]$$

where $\lambda_n(M)$ is determined by $\tau_r^{'}(M)$ with sufficiently large prime r. By the Conjecture of R. Lawsence [La] proved recently by Rozansky [R2], if $|H_1(M,Z)| \neq 0 \pmod r$ then the cyclotomic series $\sum_{n=0}^{\infty} \lambda_n^r(M)h^n$ converges r-adicly to

$$(\frac{|H_1(M,Z)|}{r})|H_1(M,Z)|\tau_r^{'}(M)$$

where $h = e^{\frac{2\pi i}{r}} - 1$. Therefore for any suitably large prime r, $\tau_r^{'}(M)$ is determined by any sequence of $\{\tau_{r_n}^{'}(M)\}$, where r_n is prime and $\lim_{n\to\infty} r_n = \infty$. While for $M = X_{\text{tref}}(0)$, we see from the formula above that this property does not hold. Since we have such a

sequence of $\{\tau'_{r_n}(M)\}$ with $\tau'_{r_n}(M) = 0$, but there exists abitrarily large prime r with $\tau'_r(M) \neq 0$. It thus seems that generalizing Ohtsuki's invariant to general 3-manifolds is impossible.

Remark 3. For lens spaces, ξ_r which is equivalent to τ'_r has been obtained in [LL1], and explicit formula for τ'_r is obtained in [LL2] in order to calculate Ohtsuki's invariant. An example is given in [LL3] to show that Ohtsuki's invariant does not determine all τ'_r.

2 Proof of Theorem 1 and its corollary

2.1 Reducing to $\xi_r(M, e_r)$

Recall from [Li] that

$$\tau'_r(M) = (\sqrt{\frac{4}{r}}\sin\frac{\pi}{r})^{\nu}\Theta_r(M, \pm ie_{4r}), \qquad 3 \leq r \equiv \pm 1 \pmod 4$$

where e_a stands for $e^{\frac{2\pi i}{a}}$, ν is the first Betti number of M , and Θ_r is defined in [BHMV]. And

$$\Theta_r(M, \pm ie_{4r}) = 2^{-\nu}\xi_r(M, \mp ie_{4r}) = 2^{-\nu}\xi_r\ (M, e_r^{\frac{1\mp r}{4}}), \qquad 3 \leq r \equiv \pm 1 \pmod 4$$

where ξ_r is defined in [Li] and the formula above is also proved there. Therefore

$$\tau'_r(M) = (\sqrt{\frac{1}{r}}\sin\frac{\pi}{r})^{\nu}\xi_r\ (M, e_r^{\frac{1\mp r}{4}})$$

Our working line is then to calculate $\xi_r(M, e_r)$ first , then use the Galois automorphism of the r-th cyclotomic field sending $e_r \to\ e_r^{\frac{1\mp r}{4}}$ to get $\xi_r(M, e_r^{\frac{1\mp r}{4}})$, hence $\tau'_r(M)$.

For a rational number α, denote by $< m_l, m_{l-1}, \ldots, m_1 >$ its continued fraction expression

$$\alpha = m_l - \cfrac{1}{m_{l-1} - \ddots - \cfrac{1}{m_2 - \cfrac{1}{m_1}}}$$

let $p_k/q_k\ =< m_{k,l_k}, m_{k,l_{k-1}}, \cdots, m_{k,l_1}\ >$, then $X(p_1/q_1, \cdots, p_n/q_n)$ is represented by the framed link:

where a dot $\bullet$ lebbelled with a number m means an unknot with framing m, and two dots connected by a line means that 2 relevant unknots form a Hopf link.

This shows that $X(p_1/q_1, \cdots, p_n/q_n)$ is a plumed manifold. By using the formula in [LL4] for Θ_r of plamed manifolds together with relation between Θ_r and ξ_r in [Li], we get

$$\xi_r(X(p_1/q_1, \cdots, p_n/q_n), e_r) = I/II$$

where

$$I = (e_r^2 - e_r^{-2})^{-(N+1)} e_r^{-\sum_{k,l} m_{k,l}} \sum_{j=1}^{r} (e_r^{2j} - e_r^{-2j})^{2-n} \prod_{k=1}^{n} S_{k,l_k}(j)$$

with N being the number of components of the framed link, i.e. $N = \sum_k l_k + 1$, and

$$S_{k,l_k}(j) = \sum_{j_1,\cdots,j_{l_k}=1}^{r} e_r^{\sum_{l=1}^{l_k} m_{k,l} j_l^2} (e_r^{2j_1} - e_r^{-2j_1})(e_r^{2j_{l_k} j} - e_r^{-2j_{l_k} j}) \prod_{t=1}^{l_k-1} (e_r^{2j_t j_{t+1}} - e_r^{-2j_t j_{t+1}})$$

and

$$II = s_+^{b^+} s_-^{b^-}, \quad s_+ = \frac{-2e_r^{-3}}{e_r^2 - e_r^{-2}} \sqrt{r} \varepsilon(r), \quad s_- = \bar{s}_+$$

with b_+ and b_- the numbers of positive and negative eigen values of the linking matrix respectively, and $\bar{s}_+$ means the complex conjugete of s_+.

2.2. Good continued franction expression.

We call $< m_l, m_{l-1}, \ldots, m_1 >$ a good continued framction expression of α, or simply a good expression of α, if $m_l = [\alpha] + 1$ with $[\alpha]$ the integer part of α, and $< m_{l-1}, \ldots, m_1 >= ([\alpha] + 1 - \alpha)^{-1}$ such that $m_1, m_2, \ldots, m_{l-1}$ are all ≥ 2 if $[\alpha] \neq \alpha$, and $< m_{l-1}, \ldots, m_1 >=< 1 >$ if $[\alpha] = \alpha$.

In [LL1], for $\alpha = p/q$ with $p > 0$, $q > 0$, and $\alpha =< m_l, m_{l-1}, \ldots, m_1 >$ with all $m_j \geq 2$, $N_{j,i}$ for $1 \leq i \leq j \leq l$ is defined to be the numerator of $< m_j, \ldots, m_i >$, then

$$N_{l,1} = p, \ \ N_{l-1,1} = q$$

Let $p_k/q_k =< m_{k,l_k}, \cdots, m_{k,1} >$ be a good expression, then for $i \leq j \leq l_k - 1$, since $< m_{k,j}, \cdots, m_{k,i} >$ is positive, we can still define $N_{k;j,i}$ as its numerator, and in the assumption of $q_k > 0$, we have $N_{k;l_k-1,1} = q_k$. Thus althrough p_k can be negative, we still have $p_k = N_{k;l_k,1}$ being the numerator of $< m_{k,l_k}, \cdots, m_{k,1} >$ and all results concerning $N_{j,i}$ are still true.

2.3 Calculation for $S_{k,l_k}(j)$

By Lemma 4.12, 4.20 and some other lemmas in §4 of [LL1], we have

$$\begin{aligned} S_{k,l_k}(j) = & \ (-2\sqrt{r}\varepsilon(r))^{l_k} \sqrt{c_k} \varepsilon(c_k) (\frac{p_k/c_k}{r/c_k})(\frac{q_k}{c_k}) \times \\ & (-1)^{\frac{r-1}{2}\frac{c_k-1}{2}} \textstyle\sum_{\pm} \chi^{k,\pm}(j) e_r^{-(p_k/c_k)' \frac{q_k}{c_k}(j \mp q_k^*)^2 - p_k^*(q_k^* \mp 2j)} \end{aligned}$$

where $q_k^* = N_{k;l_k,2}$ and $p_k^* = -N_{k;l_k-1,2}$ is a spicial choice of q_k^* and p_k^* satisfying $q_k^* q_k + p_k^* p_k = 1$.

Now

$$-(p_k/c_k)'\frac{q_k}{c_k}(j \mp q_k^*)^2 - p_k^*(q_k^* \mp 2j)$$

$$= -(p_k/c_k)'\frac{q_k}{c_k}j^2 \pm 2((p_k/c_k)'\frac{q_k q_k^*}{c_k} + p_k^*)j - (p_k/c_k)'\frac{q_k}{c_k}q_k^* q_k^* - p_k^* q_k^*$$

and

$$\begin{aligned}(p_k/c_k)'\frac{q_k q_k^*}{c_k} + p_k^* = &\quad (p_k/c_k)'\frac{1 - p_k p_k^*}{c_k} + p_k^* = \frac{1}{c_k}(p_k/c_k)' - p_k^*(1 - (r/c_k)'\frac{r}{c_k}) + p_k^* \\ = &\quad \frac{1}{c_k}(p_k/c_k)' + p_k^*(r/c_k)'\frac{r}{c_k},\end{aligned}$$

$$\begin{aligned}-(p_k/c_k)'\frac{q_k}{c_k}q_k^* q_k^* - p_k^* q_k^* = &\quad -(p_k/c_k)'\frac{q_k^*}{c_k}(1 - p_k^* p_k) - p_k^* q_k^* \\ = &\quad -(p_k/c_k)'\frac{q_k^*}{c_k} + p_k^* q_k^*(1 - (r/c_k)'\frac{r}{c_k}) - p_k^* q_k^* \\ = &\quad -(p_k/c_k)'\frac{q_k^*}{c_k} - p_k^* q_k^*(r/c_k)'\frac{r}{c_k}\end{aligned}$$

Thus

$$\begin{aligned}S_{k,l_k}(j) = &\quad (-2\sqrt{r}\varepsilon(r))^{l_k}\sqrt{c_k}\varepsilon(c_k)(\frac{p_k/c_k}{r/c_k})(\frac{q_k}{c_k})(-1)^{\frac{r-1}{2}\frac{c_k-1}{2}} e_r^{-(p_k/c_k)'\frac{q_k^*}{c_k} - p_k^* q_k^*(r/c_k)'\frac{r}{c_k}} \times \\ &\quad e_r^{-(p_k/c_k)'\frac{q_k}{c_k}j^2} \sum_\pm \chi^{k,\pm}(j) e_r^{\pm 2(\frac{1}{c_k}(p_k/c_k)' + p_k^*(r/c_k)'\frac{r}{c_k})j}\end{aligned}$$

2.4. Calculation for $\xi_r(X(p_1/q_1, \cdots, p_n/q_n), e_r)$

First by a diagonalization procedure for the quatratic form of the linnk matrix, we see that

$$b_+ + b_- = \begin{cases} N, & \text{if } \sum \frac{q_k}{p_k} \neq 0 \\ N-1, & \text{if } \sum \frac{q_k}{p_k} = 0 \end{cases}$$

$$b_- = \text{the number of the negative elements in the set } \{p_1/q_1, \cdots, p_n/q_n, -\sum \frac{q_k}{p_k}\},$$

since we use a food expression for every p_k/q_k. Thus by the assumption $q_k > 0$, we have

$$\begin{aligned} b_+ + b_- &= N - 1 + sign|H| \\ b_- &= \text{the number of the negative elements in the set } \{p_1, \cdots, p_n, -\frac{H}{P}\} \end{aligned}$$

and $(-1)^{\frac{r+1}{2}b_-} = (signP)^{\frac{r+1}{2}}(-sign\frac{H}{P} + 1 - sign|H|)^{\frac{r+1}{2}}$.

Now

$$s_+^{-b_+}s_-^{-b_-} = (e_r^2 - e_r^{-2})^{b_+ + b_-}(-2\sqrt{r})^{-b_+ - b_-}(-1)^{b_-}\varepsilon(r)^{-b_+ + b_-}$$

Thus

$$\xi_r(X(p_1/q_1, \cdots, p_n/q_n), e_r) = I/II = G_1G_2$$

where

$$G_1 = \quad e_r^{-\sum_k (p_k/c_k)'\frac{q_k^*}{c_k} - \sum_k p_k^* q_k^*(r/c_k)'\dfrac{r}{c_k}} \sum_{j=1}^r (e_r^{2j} - e_r^{-2j})^{2-n} e_r^{-\sum_k (p_k/c_k)'\dfrac{q_k}{c_k}j^2} \times$$

$$\prod_k \sum_\pm \chi^{k,\pm}(j) e_r^{\pm 2(\dfrac{1}{c_k}(p_k/c_k)' + p_k^*(r/c_k)'\dfrac{r}{c_k})j}$$

and

$$G_2 = \quad e_r^{3(b_+ - b_-) - \sum_{k,l} m_{k,l}}(e_r^2 - e_r^{-2})^{b_+ + b_- - N - 1}(-2\sqrt{r})^{-b_+ - b_-}(-1)^{b_-}\times$$

$$\varepsilon(r)^{b_- - b_+}(-2\sqrt{r}\varepsilon(r))^{\sum_k l_k}\prod_k(\sqrt{c_k}(\frac{p_k/c_k}{r/c_k})(\frac{q_k}{c_k})(-1)^{\frac{r-1}{2}\frac{c_k-1}{2}}\varepsilon(c_k))$$

Notice that

$$b_+ - b_- = -sign\frac{H}{P} + \sum_k (l_k - 1 + sign p_k)$$

and

$$3(l_k - 1 + sign p_k) - \sum_{l=1}^{l_k} m_{k,l} = -12s(q_k, p_k) + \frac{q_k + q_k^*}{p_k}$$

(cf.[KM2]), where $q_k^* = N_{k;l_k,2}$. Also we have $\sum_k l_k = N - 1$ and

$$(-1)^{b_-}\varepsilon(r)^{-(b_+ - b_-) + \sum_k l_k} = (-1)^{b_-}\varepsilon(r)^{2b_- - sign|H|} = (-1)^{\frac{r+1}{2}b_-}\varepsilon(r)^{-sign|H|}$$

So,

$$G_2 = \quad e_r^{-3sign\frac{H}{P} + \sum_k(-12s(q_k,p_k) + \frac{q_k+q_k^*}{p_k})}(e_r^2 - e_r^{-2})^{-2+sign|H|}(-2\sqrt{r}\varepsilon(r))^{-sign|H|}\times$$

$$(sign P)^{\frac{r+1}{2}}(-sign\tfrac{H}{P} + 1 - sign|H|)^{\frac{r+1}{2}}\prod_k(\sqrt{c_k}(\frac{p_k/c_k}{r/c_k})(\frac{q_k}{c_k})(-1)^{\frac{r-1}{2}\frac{c_k-1}{2}}\varepsilon(c_k))$$

and $\xi_r(M, e_r)$ is obtained.

2.5. The Galois automorphism

Consider the Galois automorphism sending e_r to $A = e_r^{\frac{1\mp r}{4}}$ for $r \equiv \pm 1 \pmod 4$. In the formula for $\xi_r(M, e_r)$, since

$$G_1 = \sum_{j=1}^r (e_r^{2j} - e_r^{-2j})^{2-n}\prod_k \sum_\pm \chi^{k,\pm}(j) e_r^{-(p_k/c_k)'\dfrac{q_k}{c_k}(j \mp q_k^*)^2 - p_k^*(q_k^* \mp 2j)}$$

and when $\chi^{k,\pm}(j) \neq 0,\ c_k|\, j \mp q_k^*$, we see that the image of G_1 under the automorphism is obtained via replacing e_r by A.

For any positive factor c of r, we have

$$\sqrt{c}\varepsilon(c) = \sum_{x=1}^{c} e_c^{x^2} = \sum_{x=1}^{c} e_r^{\frac{r}{c}x^2}$$

So the image of $\sqrt{c}\varepsilon(c)$ is

$$\sum_{x=1}^{c} e_r^{\frac{1\mp r}{4}\frac{r}{c}x^2} = \sum_{x=1}^{c} e_c^{\frac{1\mp r}{4}x^2} = \sqrt{c}\varepsilon(c)(\frac{(1\mp r)/4}{c})$$

$4 \cdot \frac{1\mp r}{4} \equiv 1 \pmod r$ implies $4 \cdot \frac{1\mp r}{4} \equiv 1 \pmod c$. This means that $\frac{1\mp r}{4} \pmod c$ is a square. Hence

$$\left(\frac{(1\mp r)/4}{c}\right) = 1$$

and the image of $\sqrt{c}\varepsilon(c)$ is $\sqrt{c}\varepsilon(c)$. Thus the image of G_2 is obtained by putting A in the place of e_r.

2.6 Independentness of the choice of q_k^* and p_k^*

So far in the formula for $\xi_r(M, e_r)$, q_k^* and p_k^* depends on the good exrpression of p_k/q_k. If q_k^* is changed, it must become $q_k^* + mp_k$ for some $m \in Z$, and then p_k^* becomes $p_k^* - mq_k$. Look at the change of G_1, we see that $\chi^{k,\pm}(j)$ does not change.
While $-(p_k/c_k)'\dfrac{q_k}{c_k}(j \mp q_k^*)^2 - p_k^*(q_k^* \mp 2j)$ changes to

$$\begin{aligned} & -(p_k/c_k)'\frac{q_k}{c_k}(j \mp q_k^* \mp mp_k)^2 - (p_k^* - mq_k)(q_k^* \mp 2j + mp_k) \\ = \quad & -(p_k/c_k)'\frac{q_k}{c_k}(j \mp q_k^*)^2 - p_k^*(q_k^* \mp 2j) + X + Y \end{aligned}$$

where

$$\begin{aligned} X \quad &= -m(p_k/c_k)'\frac{p_k}{c_k}(mp_k + 2(q_k^* \mp j))q_k \\ Y \quad &= -mp_k^*p_k - mq_k^*q_k + m^2p_kq_k + 2mq_k(q_k^* \mp 2j) \end{aligned}$$

Since $(p_k/c_k)'\dfrac{p_k}{c_k} \equiv 1 \pmod{\dfrac{c_k}{r}}$, and $mp_k + 2(q_k^* \mp j) \equiv 0 \pmod r$,

$$X \equiv -m(mp_k + 2(q_k^* \mp j))q_k \pmod r$$

Thus

$$X + Y \equiv -mp_k^*p_k - mq_k^*q_k \equiv -m \pmod r$$

and G_1 changes to $e_r^{-m}G_1$.

Now it is obvious that G_2 changes to $e_r^m G_2$. So $\xi_r(M, e_r)$ does not depends on the special choice of q_k^* and p_k^*, and the proof for Theorem 1 is complete.

2.7. Proof of the Corollary

Since

$$-12s(q_k,p_k)+\frac{q_k+q_k^*}{p_k}=3(l_k-1+signp_k)-\sum_{l=1}^{l_k}m_{k,l}$$

and

$$-12s^{\surd}(q_k,p_k)\equiv 3(l_k-1+signp_k)-\sum_{l=1}^{l_k}m_{k,l}-p_k^{'}q_k^*-p_k^{'}q_k \pmod r$$

$$\sum_k p_k^{'}q_k=P'H$$

$$\prod_k(\frac{p_k}{r})=(\frac{P}{r})=(\frac{|P|}{r})(signP)^{\frac{r-1}{2}}$$

we are done.

3 Comparision with the formula of Rozansky

Under the assumption of p_k, $q_k\not\equiv 0 \pmod r$, $H\neq 0$, and r being prime, Rozansky's formula for $\tau_r^{'}(X(p_1/q_1,p_2/q_2,\cdots,p_n/q_n))$ is

$$\tau_r^{'}=\tfrac{i}{2\sqrt{r}}\quad e^{\frac{i\pi}{4}sign\frac{H}{P}(\varepsilon(r)^2+3\frac{r-2}{r})}$$

$$\times(\tfrac{|P|}{r})signPe_r^{4'P'H-3\sum_{k=1}^{n}s^{\surd}(q_k,p_k)}(e_r^{2'}-e_r^{-2'})^{-1}$$

$$\times\textstyle\sum_{0\le\beta<2r,\beta\in 2Z+1}e_r^{-4'P'H\beta^2}(e_r^{2'\beta}-e_r^{-2'\beta})^{2-n}\prod_{k=1}^{n}(e_r^{2'p_k^{'}\beta}-e_r^{-2'p_k^{'}\beta})$$

Let $\beta=2(\alpha-2')+1$, then $\beta\equiv 2\alpha \pmod r$. And let $\alpha=2'j$, then we have

$$\begin{aligned}\textstyle\sum_{0\le\beta<2r,\beta\in 2Z+1}\quad & e_r^{-4'P'H\beta^2}(e_r^{2'\beta}-e_r^{-2'\beta})^{2-n}\textstyle\prod_{k=1}^{n}(e_r^{2'p_k^{'}\beta}-e_r^{-2'p_k^{'}\beta})\\ &=\textstyle\sum_{\alpha=1}^{r}e_r^{-P'H\alpha^2}(e_r^{\alpha}-e_r^{-\alpha})^{2-n}\prod_{k=1}^{n}(e_r^{p_k^{'}\alpha}-e_r^{-p_k^{'}\alpha})\\ &=\textstyle\sum_{j=1}^{r}(A^{2j}-A^{-2j})^{2-n}A^{-P'Hj^2}\prod_{k=1}^{n}(A^{2p_k^{'}j}-A^{-2p_k^{'}j})\end{aligned}$$

It can be checked that in his formula the term

$$ie^{\frac{i\pi}{4}sign\frac{H}{P}(\varepsilon(r)^2+3\frac{r-2}{r})}=(sign\frac{H}{P})^{\frac{r+1}{2}}\varepsilon(r)A^{-3sign\frac{H}{P}}$$

And in our formula the term

$$\begin{aligned}(-\varepsilon(r))^{-sign|H|}(-sign\tfrac{H}{P}+1-sign|H|)^{\frac{r+1}{2}}\quad &=-\varepsilon(r)(-1)^{\frac{r-1}{2}}(-sign\tfrac{H}{P})^{\frac{r+1}{2}}\\ &=\varepsilon(r)(sign\tfrac{H}{P})^{\frac{r+1}{2}}\end{aligned}$$

Hence two formulas coincide.

4 Proof of Theorem 2

Theorem 2 is equivalent to algebraic integrality of $\xi_r(X(p_1/q_1,\cdots,p_n/q_n),e_r)$, when r is coprime to at least $n-2$ of p_k,

We assume $c_k=1$ for $3\le k\le n$. Then

$$\xi_r(M,e_r)=(2\varepsilon(r)\sqrt{r})^{-sign|H|}(e_r^2-e_r^{-2})^{-2+sign|H|}\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}F_1F_2$$

where F_1 is an algebraic integer, and

$$F_2=\sum_{j=1}^{r}(K_+(j)+K_-(j))K(j)$$

with

$$K_\pm(j)=\chi^{1,\pm}(j)e_r^{-(p_1/c_1)'\frac{q_1}{c_1}(j\mp q_1^*)^2\pm 2p_1^*(j\mp q_1^*)}$$

and

$$\begin{aligned}K(j)=\;&(\textstyle\sum_\pm \chi^{2,\pm}(j)e_r^{-(p_2/c_2)'\frac{q_2}{c_2}(j\mp q_2^*)^2\pm 2p_2^*(j\mp q_2^*)})\times\\ &e_r^{(-\sum_{k=3}^n p_k' q_k)j^2}(e_r^{2j}-e_r^{-2j})^{2-n}\textstyle\prod_{k=3}^n(e_r^{2p_k'j}-e_r^{-2p_k'j})\end{aligned}$$

Notice that $K(j)$ and $K_\pm(j)$ are all functions of j with period r, and

$$\chi^{k,\pm}(-j)=-\chi^{k,\mp}(j)$$

Hence

$$\begin{aligned}\textstyle\sum_{j=1}^r K_+(j)K(j)\;&=\textstyle\sum_{j=1}^r K_+(-j)K(-j)\\ &=\textstyle\sum_{j=1}^r(-\chi^{1,-}(j)(\sum_\pm -\chi^{2,\mp}(j)e_r^{-(p_2/c_2)'\frac{q_2}{c_2}(j\pm q_2^*)^2\mp 2p_2^*(j\pm q_2^*)})\\ &\times e_r^{(-\sum_{k=3}^n p_k' q_k)j^2}(e_r^{2j}-e_r^{-2j})^{2-n}\textstyle\prod_{k=3}^n(e_r^{2p_k'j}-e_r^{-2p_k'j}))\\ &=\textstyle\sum_{j=1}^r K_-(j)K(j)\end{aligned}$$

and

$$F_2=2\sum_{j=1}^{r}K_+(j)K(j)$$

Let

$$\begin{aligned}F_2^\pm=\;&2\textstyle\sum_{j=1}^r K_+(j)\chi^{2,\pm}(j)e_r^{-(p_2/c_2)'\frac{q_2}{c_2}(j\mp q_2^*)^2\pm 2p_2^*(j\mp q_2^*)}\\ &\times e_r^{(-\sum_{k=3}^n p_k' q_k)j^2}(e_r^{2j}-e_r^{-2j})^{2-n}\textstyle\prod_{k=3}^n(e_r^{2p_k'j}-e_r^{-2p_k'j})\end{aligned}$$

Consider the set

$$S^{\pm} = \{j/\ \chi^{1,+}(j)\chi^{2,\pm}(j) \neq 0\} \subset Z_r$$

If S^η is empty, then $F_2^\eta = 0$, where $\eta = +$ or $-$. Now assume $S^\eta \neq \emptyset$, and let c be the least common multiple of c_1 and c_2. Then $c|r$, and S^η is a residue class of $Z_r \pmod{Z_c}$, i.e.

$$S^\eta = \{\, a_\eta + xc/x \in Z_{\frac{r}{c}}\}$$

for some $a_\eta \in Z_r$.

Since

$$(e_r^{2j} - e_r^{-2j})^{2-n} \prod_{k=3}^{n} (e_r^{2p_k' j} - e_r^{-2p_k' j}) = \sum_{s=1}^{m} e_r^{2b_s j}$$

for some integers $b_1, \cdots, b_m$, we see that substituting j with $a_\eta + xc$ yields

$$F_2^\eta = 2\eta \sum_{s=1}^{m} (e_r^{\beta_{\eta,s}} \sum_{x=1}^{r/c} e_r^{c(Dx^2 + E_{\eta,s}x)})$$

where $\beta_{\eta,s}$, $E_{\eta,s}$ are integers depending on η ans s, while D is an integer independent of η ans s.

Let $d = (D, r/c)$, we have by Theorem 2.1 and 2.2 in [LL1]

$$\sum_{x=1}^{r/c} e_{r/c}^{Dx^2 + E_{\eta,s}x} = \begin{cases} r/c, & \text{if } D = 0, \text{ and } \frac{r}{c}|E_{\eta,s} \\ 0, & \text{if } D = 0, \text{ and } \frac{r}{c} \nmid E_{\eta,s} \\ & \text{or } D \neq 0 \text{ and } d \nmid E_{\eta,s} \\ d\sum_{x=1}^{r/cd} e_{r/cd}^{\frac{D}{d}x^2 + \frac{E_{\eta,s}}{d}x} = \pm d\varepsilon(r/cd)\sqrt{r/cd}\, e_r^{F_{\eta,s}}, & \text{if } D \neq 0 \text{ and } d|E_{\eta,s} \end{cases}$$

where $F_{\eta,s}$ is an integer. Thus

$$F_2^\eta = \begin{cases} 2\frac{r}{c}c^\eta, & \text{if } D = 0 \\ 2d\varepsilon(r/cd)\sqrt{r/cd}\, c^\eta, & \text{if } D \neq 0 \end{cases}$$

for some algebraic integer c^η, and we have the following formula which is true always:

$$(*)\ \ \xi_r(M, e_r) = (2\varepsilon(r)\sqrt{r})^{-sign|H|}(e_r^2 - e_r^{-2})^{-2+sign|H|} \times \begin{cases} 2\frac{r}{c}\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}G, & \text{if } D = 0 \\ 2\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}d\varepsilon(r/cd)\sqrt{\frac{r}{cd}}G, & \text{if } D \neq 0 \end{cases}$$

for some algebraic integer G

Case 1. $(c_1, c_2) > 1$. By Lemma 4.16 in [LL2], it appears that

$$(\varepsilon(r)\sqrt{r})^{-1}\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}\frac{r}{c} = \pm\varepsilon(c_1c_2/c)\sqrt{\frac{c_1c_2}{c}}\varepsilon(r_c)\sqrt{\frac{r}{c}}$$

and

$$\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}\varepsilon(r/_{cd})\sqrt{\frac{r}{cd}}d = \pm\varepsilon(c_1c_2/c)\sqrt{\frac{c_1c_2}{c}}\varepsilon(r)\sqrt{r}\varepsilon(d)\sqrt{d}$$

Since $(c_1, c_2) > 1$, we have $c_1 > 1, c_2 > 1$ and $\frac{c_1c_2}{c} > 1$. Then by Corollary 5.4 in [LL1],

$$\frac{\varepsilon(c_i)\sqrt{c_i}}{e_r^2 - e_r^{-2}},\ i = 1, 2,\ \frac{\varepsilon(c_1c_2/c)\sqrt{\frac{c_1c_2}{c}}}{e_r^2 - e_r^{-2}} \text{ and } \varepsilon(b)\sqrt{b} \text{ for some positive factor } b \text{ of } r$$

are all algebraic integers, so is $\xi_r(M, e_r)$ by $(*)$.

Case 2. $(c_1, c_2) = 1$. In this case, both S^+ and S^- are nonempty.

Case 2.1. $D = 0$. If $\frac{r}{c} > 1$, since $\frac{r}{c} = \pm(\sqrt{\frac{r}{c}}\varepsilon(r/c))^2$, we are done by $(*)$ and the fact $(e_r^2 - e_r^{-2})^{-1}\sqrt{\frac{r}{c}}\varepsilon(r/c)$ being an algebraic integer.

If $r = c$, then $\sum_{x=1}^{r/c} e_{r/c}^{c(Dx^2+E_{\eta,s}n)} = 1$, and

$$F_2^+ + F_2^- = 2\sum_{s=1}^{m}(e_r^{\beta_{+,s}} - e_r^{\beta_{-,s}})$$

Therefore

$$\xi_r(M, e_r) = (2\varepsilon(r)\sqrt{r})^{-sign|H|}(e_r^2 - e_r^{-2})^{-2+sign|H|}\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2}2\sum_{s=1}^{m}(e_r^{\beta_{+,s}} - e_r^{\beta_{-,s}})$$

Since for any $a, b \in Z, e_r^a - e_r^b$ can be written as $e_r^u(e_r^{2v} - e_r^{-2v})$ for some $u, v \in Z$, and $c = r > 1$ implies that one of c_1 and c_2 must > 1, we are done for the case of $H = 0$. If $H \neq 0$, we have

$$\begin{aligned}(\varepsilon(r)\sqrt{r})^{-1}\varepsilon(c_1)\sqrt{c_1}\varepsilon(c_2)\sqrt{c_2} &= \pm(\varepsilon(c)\sqrt{c})^{-1}\varepsilon(c_1c_2)\sqrt{c_1c_2} = \pm\varepsilon(c_1c_2/c)\sqrt{\tfrac{c_1c_2}{c}}\\ &= \pm 1\end{aligned}$$

It is done again.

Case 2.2. $D \neq 0$. Then by a formula in Case 1,

$$\begin{aligned}\xi_r(M, e_r) = \ &\pm(2\varepsilon(r)\sqrt{r})^{-sign|H|}(e_r^2 - e_r^{-2})^{-2+sign|H|}\varepsilon(c_1c_2/c)\sqrt{\frac{c_1c_2}{c}}\\ &\varepsilon(r)\sqrt{r}\varepsilon(d)\sqrt{d}\textstyle\sum_{s=1}^{m}(e_r^{\beta_{+,s}-F_{+,s}} - e_r^{\beta_{-,s}-F_{-,s}})\end{aligned}$$

Writing $e_r^a - e_r^b$ as $e_r^u(e_r^{2\nu} - e_r^{-2\nu})$, for some $u, v \in Z$ again, we are done.

Notice that in the case $H = 0$, $\nu(M) = 1$, and $\xi_r(M, e_r) = 2$ times an algebraic integer. So $\Theta_r(M, e_{2r}) = 2^{-\nu(M)}\xi_r(M, -e_{2r})$ is an algebraic integer. Theorem 2 and Remark 2 are proved.

5 Calculation for $X_{tref}(0)$

We have by the Corollary of Theorem 1 that if $3 \not| r$, then

$$\begin{aligned} \xi_r(X(3/_1, 6/_1, -2/_1), e_r) = \ & e_r^{-12(s^{\checkmark}(1,3)+s^{\checkmark}(1,6))}(e_r^2 - e_r^{-2})^{-2} \times \\ & \textstyle\sum_{j=1}^r ((e_r^{2\cdot 3'j} - e_r^{-2\cdot 3'j})(e_r^{2\cdot 6'j} - e_r^{-2\cdot 6'j}) \sum_{k=0}^{2'-1} e_r^{2j(2'-1-2k)}) \end{aligned}$$

since $H = 0$, $|P| = 6^2$, and $s(1,-2) = 0$. Expand the term $\sum_{j=1}^r$ to

$$\begin{aligned} \textstyle\sum_{k=0}^{2'-1}\sum_{j=1}^r (e_r^{2j(3'+6'+2'-1-2k)} \quad & -e_r^{2j(3'-6'+2'-1-2k)} \\ & -e_r^{2j(-3'+6'+2'-1-2k)} + e_r^{2j(-3'-6'+2'-1-2k)}) \end{aligned}$$

We should look for the solutions of following equations for $0 \le k \le 2' - 1 = \dfrac{r-1}{2}$:

$$\begin{array}{ll} (1)\ 2(3' + 6' + 2' - 1 - 2k) \equiv 0 & \pmod r \\ (2)\ 2(3' - 6' + 2' - 1 - 2k) \equiv 0 & \pmod r \\ (3)\ 2(-3' + 6' + 2' - 1 - 2k) \equiv 0 & \pmod r \\ (4)\ 2(-3' - 6' + 2' - 1 - 2k) \equiv 0 & \pmod r \end{array}$$

Multiplying the equations by 3, they become

$$\begin{array}{ll} (1)\ \ -12k \equiv 0 & \pmod r \\ (2)\ \ -2 - 12k \equiv 0 & \pmod r \\ (3)\ \ -4 - 12k \equiv 0 & \pmod r \\ (4)\ \ -6 - 12k \equiv 0 & \pmod r \end{array}$$

Thus (1) has just one solution $k = 0$. For (2), $-6k = 1 \pmod r$. We should look for the solutions of l satisfying $1 \le 2l + 1 \le r$ and

$$\frac{r - (2l + 1)}{2} \times (-6) \equiv 1$$

i.e. $3(2l + 1) = 1 + nr$. Then n can be only 2, and only when $r \equiv 1 \pmod 3$, there is a solution. For (3), we have the same conclusion as for (2). For (4), $2k \equiv -1$ has a unique solution $k = \frac{r-1}{2}$.

$$-12s(1,3) = \frac{-2}{3} \quad \text{and} \quad -12s(1,6) = \frac{-10}{3}$$

So

$$\xi_r(X_{tref}(0), e_r) = \begin{cases} 0, & \text{if } r \equiv 1 \pmod 3 \\ e_r^{-4}\dfrac{2r}{(e_r^2 - e_r^{-2})^2}, & \text{if } r \equiv -1 \pmod 3 \end{cases}$$

and

$$\tau_r'(X_{tref}(0)) = \begin{cases} 0 & \text{if } r \equiv 1 \pmod 3 \\ \sqrt{\dfrac{1}{r}}\sin\dfrac{\pi}{r}\dfrac{2r}{(e_r^{2'} - e_r^{-2'})^2}e_r^{-1} = \dfrac{-\sqrt{r}}{2\sin\frac{\pi}{r}}e_r^{-1}, & \text{if } r \equiv -1 \pmod 3 \end{cases}$$

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